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INTEGRAL SOLUTIONS OF TERNARY QUINTIC DIOPHANTINE EQUATION $x^2 + y^2 = 10z^5$

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ABSTRACT:

The Diophantine equation of degree five with three unknowns represented by $x^2 + y^2 = 10z^5$ is investigated for its non-zero distinct integral solutions. A few interesting relations between the values of x, y, z and special numbers are presented.

KEYWORDS:

Ternary Quintic equation, Integral solutions, Polygonal numbers.

NOTATIONS:

$t_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right] =$ Polygonal number of rank n with sides m

$P_n^r = \frac{1}{6}n(n+1)[(r-2)n + (5-r)] =$ Pyramidal number of rank n with sides r

$P_n^r = n(n+1) =$ Pronic number of rank n

$SO_n = n(2n^2 - 1) =$ Stella Octangula number

$Ct_{m,n} = \frac{mn(n+1)+2}{2} =$ Centered Polygonal number of rank n with sides m

$OH_n = \frac{1}{3}n(2n^2 + 1) =$ Octahedral number

$Pt_n = \frac{n(n+1)(n+2)(n+3)}{24}$ = Pentatope number of rank n

$gn_a = 2a - 1$ = Gnomonic number of rank a

$HauyOH_n = \frac{1}{3}(2n-1)(2n^2 - 2n + 3)$ = Hauy Octahedral number

INTRODUCTION:

Number theory is a branch of pure mathematics that deals with the properties and relationships of numbers, especially integers. It explores topics like prime numbers, divisibility, congruences, and the solutions to equations involving whole numbers. Some well-known problems in number theory include finding prime numbers, understanding the distribution of primes, and solving Diophantine equations. It has applications in cryptography, computer science, and other fields. The non-homogeneous ternary Quintic Diophantine equation offers an unlimited field for research due to their variety [1-7]. A Diophantine equation is an equation that seeks integer solutions. Specifically, it involves finding integer values for variables that satisfy the equation. These equations can involve one or more variables, and the goal is to find integer solutions, as opposed to real or complex solutions. A quadratic equation is a polynomial equation of the form $ax^2 + bx + c = 0$ where a, b and c are constants and z represents the variable or unknown. A lot is known about equations in two variables in higher degrees. For equations with more than three variables and degree at least three, very little is known. It is worth to note that undesirability appears in equations, even perhaps at degree four with small coefficients. One may refer [8-12] quintic Diophantine equations with three unknowns. Thus, in this communication a fifth degree non-homogeneous Diophantine equation with three unknowns represented by $x^2 + y^2 = 10z^5$ is analyzed for its non-zero integralsolutions. A few interesting relations between the values of x, y, z and special numbers are presented.

METHOD OF ANALYSIS

PROBLEM:

The fifth degree Diophantine equation with three unknowns to be solved is

$$x^2 + y^2 = 10z^5 \qquad (1)$$

Three different patterns of solutions of (1) are illustrated below:

Pattern 1:

Choosing z as $z = a^2 + b^2$ and write equation (1) in factorizable from as follows:

$$(x+iy)(x-iy) = (3+i)(3-i)(a+ib)^5(a-ib)^5 = \delta(constant)$$

$$\frac{(x+iy)}{(3+i)(a+ib)^5} = \frac{(3-i)(a-ib)^5}{(x-iy)} = \delta$$

Setting $(x+iy) = \delta(3+i)(a+ib)^5$

$$(x+iy) = \delta(3+i)(a^5 + (ib)^5 + 10a^2(ib)^3 + 10a^3(ib)^2 + 5a(ib)^4 + 5a^4(ib))$$

$$= \delta(3+i)(a^5 + ib^5 - i10a^2b^3 - 10a^3b^2 + 5ab^4 + i5a^4b)$$

$$= 3\delta a^5 + i3\delta b^5 - i30\delta a^2b^3 - 30\delta a^3b^2 + 15\delta ab^4 + i15\delta a^4b + i\delta a^5 - \delta b^5 + 10\delta a^2b^3 - i10\delta a^3b^2 + i5\delta ab^4 - 5\delta a^4b$$

Equating real and imaginary parts

$$x(a,b) = 3\delta a^5 - 30\delta a^3b^2 + 15\delta ab^4 + 10\delta a^2b^3 - 5\delta a^4b - \delta b^5$$

$$y(a,b) = \delta a^5 - 30\delta a^2b^3 + 15\delta a^4b - 10\delta a^3b^2 + 5\delta ab^4 + 3\delta b^5$$

Let δ=1,

$$x(a,b) = 3a^5 - 30a^3b^2 + 15ab^4 + 10a^2b^3 - 5a^4b - b^5$$

$$y(a,b) = a^5 - 30a^2b^3 + 15a^4b - 10a^3b^2 + 5ab^4 + 3b^5$$

$$z = a^2 + b^2$$

The above solutions x, y and z satisfies the following properties

1) $x(1,b) + y(1,b) - 4t_{5,b} - 12P_b^5 \equiv 0(mod\ 2)$
2) $x(a,3) + 90Pr_a + 54p_a^5 + 27OH_a \equiv 0(mod\ 3)$
3) $x(a,1) + y(a,1) - 14Pr_a - 12P_a^5 + 24OH_a \equiv 0(mod\ 2)$
4) $y(5,b) - 2t_{3,b} * Pr_b + t_{8,b} * t_{6,b} - 3Pr_b^2 + 6P_b^5 \equiv 2(mod3)$
5) $x(1,a) - 12P_a^5 * Pr_a + 63OH_a^2 + t_{6,a} * t_{6,a} - 2t_{8,a} - 3gn_a \equiv 0(mod7)$
6) $x(a,1) + y(a,1) - Ct_{a,2}^2 * 10Pr_a - 5So_a + 6Pr_a + gn_a + 1 \equiv 0(mod4)$

Pattern 2:

Equation (1) can also be written in the factorable form as

$$(x+iy)(x-iy)=(1+3i)(1-3i)\ [(a+ib)(a-ib)]^5$$

Setting $(x+iy)=(1+3i)(a+ib)^5$

$$(x+iy)=(1+3i)(a^5+(ib)^5+10a^2(ib)^3+10a^3(ib)^2+5a(ib)^4+5a^4(ib))$$

$$=(1+3i)(a^5+ib^5-i10a^2b^3-10a^3b^2+5ab^4+i5a^4b)$$

$$=a^5+ib^5-i10a^2b^3-10a^3b^2+5ab^4+i5a^4b+i3a^5-3b^5+$$
$$30a^2b^3-i30a^3b^2+i15ab^4-15a^4b$$

Equating real and imaginary parts

$$x(a,b)=a^5-10a^3b^2+5ab^4+30a^2b^3-15a^4b-3b^5$$

$$y(a,b)=3a^5-10a^2b^3+5a^4b-30a^3b^2+15ab^4-b^5$$

$$z(a,b)=a^2+b^2$$

The above solutions x, y and z satisfies the following properties:

1) $x(a,1)+y(a,1)-15HauyOH_a+10Pr_a^2\equiv 1(mod\ 4)$
2) $x(a,1)+72Pt_a+2So_a+6P_a^4*Pr_a+2t_{3,a}+Pr_a^2+gn_a\equiv 1(mod\ 3)$
3) $y(a,1)+24Pt_a+30P_a^5*Pr_a+t_{8,a}*t_{6,a}+15OH_a+t_{6,a}+Pr_a+5\equiv 0(mod 6)$
4) $y(1,b)+42P_b^5*Pr_b+2So_b+9Pr_b^2-t_{5,b}+10gn_b-4\equiv 0(mod 11)$
5) $x(a,1)+y(a,1)+12P_a^5*Pr_a+8So_a+gn_a-7\equiv 0(mod 10)$
6) $5x(a,1)-10So_a+30HauyOH_a-5gn_a+5Ct_{2,a}\equiv 0(mod 5)$

Pattern 3:

Introducing the transformation

$x=10^3X\,,\quad y=10^3Y\quad$ and $\quad z=10W$ (2)

We get $\quad X^2+Y^2=W^5$ (3)

Choosing $W=A^2+B^2$ and employing the method of factorization, (3) is expressed as a system of double equations given by

$$(X+iY)=(A+iB)^5$$

$$(X - iY) = (A - iB)^5$$

Equating real and imaginary in either of the above two equations the values of X and Y are given by

$$X(a,b) = A^5 - 10A^3B^2 + 5AB^4$$

$$Y(a,b) = 5A^4B - 10A^2B^3 - B^5$$

In view of (2), the non-zero integral solutions of (1) are given by

$$x(A,B) = 10^3(A^5 - 10A^3B^2 + 5AB^4)$$

$$y(A,B) = 10^3(5A^4B - 10A^2B^3 - B^5)$$

$$Z(A,B) = 10(A^2 + B^2)$$

For simplicity and clear understanding a few interesting properties satisfied by x, y and z are presented below:

1) $x(1,B) + y(1,B) + 150OH_B + 100SO_B \equiv 0(mod\ 25)$
2) $x(1,B) + y(1,B) + 150OH_B + 100SO_B \equiv 0(mod\ 10)$
3) $x(A,1) + 30HauyOH_A + 10Ct_{A,1} + 100Pr_A \equiv 0(mod\ 10)$
4) $x(A,1) + 30HauyOH_A + 10Ct_{A,1} + 100Pr_A \equiv 0(mod\ 2)$
5) $Y(B,1) - 20SO_B + 20gn_B + 20Pr_B \equiv 0(mod\ 20)$
6) $x(1,B) + 16t_{3,B} + 8gn_B \equiv 0(mod\ 4)$
7) $x(1,A) + 16t_{3,A} - 8gn_A + 4SO_A \equiv 0(mod\ 4)$
8) $x(A,1) + Y(A,1) + 2t_{A,2} + 12P_A^5 - 1 \equiv 0(mod\ 2)$
9) $x(1,A) + 16t_{3,A} - 8gn_A + 2SO_A \equiv 0(mod\ 2)$

Conclusion:

In this paper, we have presented three different methods of obtaining non-zero distinct integer solutions of the non-homogeneous Diophantine equation $x^2 + y^2 = 10z^5$. To conclude one may search for other choices of non-homogeneous ternary quintic Diophantine equations along with solutions and their corresponding properties.

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